

Dynamical instability in kicked Bose-Einstein condensates

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Bose-Einstein condensates subject to short pulses ('kicks') from standing waves of light represent a nonlinear analogue of the well-known chaos paradigm, the quantum kicked rotor. Previous studies of the onset of dynamical instability (ie exponential proliferation of non-condensate particles) suggested that the transition to instability might be associated with a transition to chaos. Here we conclude instead that instability is due to resonant driving of Bogoliubov modes. We investigate the Bogoliubov spectrum for both the quantum kicked rotor (QKR) and a variant, the double kicked rotor (QKR-2). We present an analytical model, valid in the limit of weak impulses which correctly gives the scaling properties of the resonances and yields good agreement with mean-field numerics.

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INTRODUCTION

The production of Bose-Einstein condensates (BECs) in dilute atomic gases has opened up a new domain for research in quantum dynamics, since BECs are intrinsically phase-coherent and can be controlled experimentally to an extremely high degree of precision [1]. An increasingly interesting aspect of the dynamics of BECs is that they represent a new arena for investigation of the interaction between nonlinearity and quantum dynamics, including quantum chaos [2, 3, 4, 5, 6, 7, 8, 9].

A BEC subject to periodic short pulses, or kicks, from standing waves of light represents a nonlinear generalization of the well-known chaos paradigm, the quantum kicked rotor (QKR). The QKR has been realized using (non-condensed) cold atoms, permitting experimental investigation of a range of interesting chaos phenomena [10]. The regime where the kick-period T is a rational multiple of π has also proved of particular interest: several studies have investigated the dynamics here with or without linearity [8, 11, 12, 13]. A number of experimental studies have also investigated kicked BECs [14]. Ensuring dynamical stability of the condensate is thus very important in studies of its coherent dynamics: if the condensate is dynamically unstable, numbers of non-condensate particles grow exponentially. If it is stable, they grow more slowly (polynomially). More broadly, the study of different types of instability in static [15] and driven BECs [16] is of much current interest.

Previous work on kicked systems [3, 6, 7] considered the onset of dynamical instability and investigated the relation with classical chaos. In [3], the possibility that instability is related to chaos in the one-body limit was investigated for the Kicked Harmonic Oscillator. In [6, 7] the correlation between chaos in the mean-field dynamics, rather, and the onset of dynamical instability, was investigated. An "instability border", determined by the kick strength K and the nonlinearity g was mapped out;

it was then found [7] that the parameter ranges for this border corresponds closely to a transition from regular to chaotic motion, of an effective classical Hamiltonian derived from the mean-field dynamics. Hence, present understanding of onset of dynamical instability in kicked BECs suggests that it may somehow be related to a transition to chaos.

In this work, we conclude that a quite different mechanism is primarily responsible for dynamical instability in the QKR-BEC. Our key finding is that it is the strong resonant driving of certain condensate modes by the kicking, which triggers loss of stability of the condensate. This mechanism is unrelated to the transition to chaos, but is rather an example of parametric resonance. In another context, the relationship between parametric resonance and dynamical instability of a BEC in a time-modulated trap is a topic of much current theoretical [16, 18] and experimental interest [19]. But to date, "Bogoliubov spectroscopy" in the analogous time-dependent system, the δ -kicked BEC, has not been investigated. Our study shows that the temporally kicked BECs open up many new possibilities in this arena.

We find that in general, for the kicked-BEC, there is no single stability border: typically, for moderate K , the condensate restabilizes just above the stability border. For small K and g the number of non-condensed atoms $N_{ex}(t)$ grows exponentially only very close to a few, isolated resonance peaks. With increasing K and g , the number of resonances which can be strongly excited by the kicking proliferates and overlaps. Our calculations show this is associated with generalized exponential instability; however this regime is, to a large degree, beyond the scope of our methods. For lower K and g , though, we introduce a simple perturbative model which provides the approximate position and width of the important resonances for both rational and irrational T .

A key finding is that, for the integer values of $T/\pi = m$ (where m is integer) values, the focus of the study in

[6], the onset of instability can occur at nonlinearities much lower than those required to resonantly excite even the very lowest collective mode – a key reason why the mechanism of parametric resonance may so far been overlooked in respect of destabilization of kicked BECs. Our model demonstrates that for this case, resonant excitation involves *two* excited modes in addition to the initial ground state mode. Hence we can explain the position of the critical stability border found in [6, 17].

We investigate both the usual QKR-BEC as well as a simple modification, obtained by applying a series of pairs of closely-spaced opposing kicks (the QKR2-BEC). This modifies substantially the relative strengths of the resonances, and provides the added novelty that the lowest modes are excited by an effective imaginary kick-strength. It is closely related to the double-kicked quantum rotor, investigated in cold atoms experiments and theory [20]. We introduce a simple analytical model based on the properties of the unperturbed condensate, which gives the distinctive properties and scaling behavior of the condensate oscillations on and off resonance.

In Section II we introduce briefly the kicked and double-kicked BEC systems. In Section III we introduce the time-dependent Bogoliubov method proposed by Castin and Dum and present numerics for the growth of non-condensate atoms. In Section IV we introduce a simple perturbative model, based on the one period time evolution operator for a kicked BEC. In Section V we show that the simple model and the time-dependent Bogoliubov numerics give excellent agreement in the limit of weak kicks. In Section VI we consider the case $T = 2\pi$ with both numerics and the perturbative model and show that the instability border found in [6, 7] is due to a novel type of compound Bogoliubov resonance.

KICKED BEC SYSTEMS

As in [6], we consider a BEC confined in a ring-shaped trap of radius R . We assume that the lateral dimension r of the trap is much smaller than its circumference, and thus we are dealing with an effectively 1D system [21]. The dynamics of the condensate wavefunction at temperatures well below the transition temperature are then governed by the 1D Gross-Pitaevskii (GP) Hamiltonian with an additional kicking potential:

$$H = H_{GP} + K \cos \theta f(t), \quad (1)$$

where

$$H_{GP} = -\frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial \theta^2} + g|\psi(\theta, t)|^2. \quad (2)$$

The short-range interactions between the atoms in the condensate are described by a mean-field term with strength $g = 8N_{tot}a_S R/r^2$, where a_S is the s-wave scattering length, and N_{tot} is the total number of atoms. For

the QKR-BEC system, $f(t) = \sum_n \delta(t - nT)$, while for the QKR2-BEC,

$$f(t) = \sum_n [\delta(t - nT) - \delta(t - nT + \epsilon)], \quad (3)$$

where T is the total period of the driving; $\epsilon \ll T$ and thus the second kick nearly cancels the first.

Experimental and theoretical studies of the double-kicked rotor [20] have shown that its quantum behavior is largely determined by an effective kick strength $K_\epsilon = K\epsilon$, provided $T \gg \epsilon$. Here we take $\epsilon = 1/25$. Hence, while for the QKR-BEC, the value $K = 1$ represents a relatively large impulse for a kicked BEC, for a double kicked BEC, $K = 1$ in the numerics below corresponds to $K_\epsilon = 0.04$, and represents only a very weak impulse. The reason for this is the near cancellation of consecutive kicks in each pair.

This mechanism has certain analogies with the so-called “quantum antiresonance” investigated in [6]: for QKR’s kicked at $T = 2\pi$, consecutive kicks effectively cancel. This means that even large values of $K \simeq 1$ and $g > 1$ represent only weak driving; for example, the instability border was found by [6] to occur at $g \simeq 2$ and $K = 0.8$.

TIME-DEPENDENT BOGOLIUBOV METHOD

The number of non-condensed atoms were calculated by making the usual Bogoliubov approximation, and following the formalism of Castin and Dum [22]. This adaptation of the Bogoliubov linearization for time-dependent potentials has been used in all studies to date of the dynamical stability of kicked condensates [3, 6, 13, 17]. The mean number of non-condensed atoms at zero temperature is given by $N_{ex}(t) = \sum_{k=1}^{\infty} \langle v_k(t) | v_k(t) \rangle$, where the amplitudes (u_k, v_k) of the Bogoliubov quasiparticle operators are governed by the coupled equations

$$i\hbar \frac{d}{dt} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} H + gQ|\psi|^2Q & gQ\psi^2Q^* \\ -gQ^*\psi^{*2}Q & -H - gQ^*|\psi|^2Q^* \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix}. \quad (4)$$

In this expression, $Q = I - |\psi\rangle\langle\psi|$ are projection operators that orthogonalize the quasiparticle modes with respect to the condensate [22]. We assume that at time $t = 0$, we have a homogeneous condensate $\psi_0 = 1/\sqrt{2\pi}$. Further discussion of the theory is given in [23].

The regime of validity of the method is discussed in [22]. The method is valid in the weakly interacting limit $1 \gg a_s^3\rho$ where ρ is the density. A limit is identified where this condition is satisfied, if one works with a constant $g \propto N_{tot}a_s$; thus the limit $a_s \rightarrow 0$ corresponds to $N_{tot} \rightarrow \infty$. A further requirement is that condensate depletion remains negligible. This condition fails after a few kicks in exponentially unstable regions. Here the method is

employed only to identify the parameter range for the onset of instability. We cut-off our calculations for $N_{ex} > 10^3$ (a reasonable threshold for small depletion in a condensate with $N_{tot} \sim 10^5$).

In Figs.1 (a) and (b) we show the number of non-condensed atoms, $N_{ex}(t = NT)$, calculated from the Bogoliubov equations (4) after $N = 200$. For small $K = 0.2$, $g = 1$, a single resonance is seen at $T \simeq 10$. For small K , resonances occur whenever the resonance condition [16] $\omega_0 + \omega_l = \omega_l \approx \frac{2n\pi}{T}$ is satisfied, where $n = 1, 2, 3..$ is an integer and ω_l is the eigenfrequency of the l -th collective mode. For larger $K = 1$, the figure shows that resonances are extremely dense and overlap with each other (and we show the behavior in this regime for $T < 10$). For overlapping resonances, unambiguous identification of each resonance is no longer possible. The key point here, however, is that in the stable regions outside the resonances, N_{ex} remains very small even after prolonged kicking.

Fig.1(b) shows oscillations of N_{ex} , as a function of time, for weak $K = 0.2$, $g = 1$, close to the isolated resonance at $T \approx 10$. The three possible regimes of: (non-resonant) weak quasi-periodic oscillations in time; (near-resonant) slower, large periodic oscillations; and (resonant) exponential growth are illustrated. The condensate energy, $E(N) = \int_0^{2\pi} d\theta \psi^*(N) (-\frac{1}{2} \frac{\partial^2}{\partial \theta^2} + \frac{g}{2} |\psi(N)|^2) \psi(N)$ after N kicks, obtained from the GPE itself, is also shown, for comparison, in the inset: at resonance, large oscillations are also seen.

Fig.2 shows the corresponding behavior for the double-kicked BEC, but now as a function of g , keeping $T = 2$, $\epsilon = 1/25$ constant and $K = 1$ or $K = 5$ (hence $K_\epsilon = 0.04$ or 0.2). The curve $K_\epsilon = 0.04$ corresponds to weak impulses and shows two isolated Bogoliubov resonances. While values of $g \simeq 10$ are large compared with current experimental values of $g \sim 0.5$ (see discussion of experimental g in [13]), resonances at small $g \sim 1$ more suitable for experimental spectroscopy can be excited by considering larger T . The curve $K_\epsilon = 0.2$ is in the overlapping resonance regime, so produces generalised instability.

In order to understand the behavior at the resonances, we introduce in Section II a model for the time evolution of perturbations from the kicked condensate, based on the usual linearization with respect to small perturbations.

II: KICKED CONDENSATE MODEL

The time-evolution of small perturbations of the condensate itself are described by an equation similar to Eq.4, see [22]. We write the condensate wavefunction in the form $\psi = \psi_0 + \delta\psi$, where ψ_0 is the unperturbed

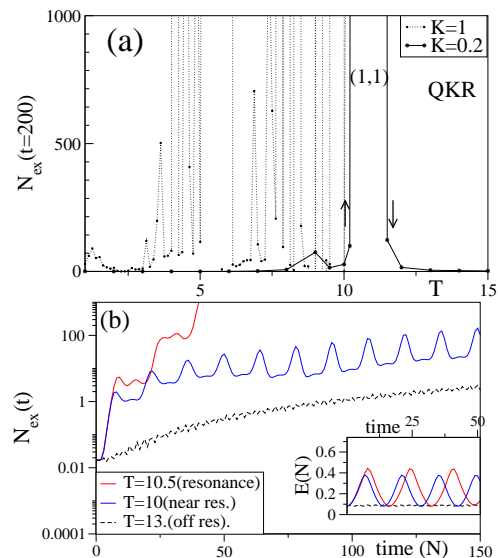


FIG. 1: (a) Shows that for weak kicks (solid line), instability occurs only at one isolated Bogoliubov (1, 1) resonance, where (n, l) denotes the n -th resonance of eigenmode l . “Up” arrows indicate onset of exponential instability; “down” arrows means stability is regained. $g = 1$. The total number of non-condensate atoms generated after 200 kicks, $N_{ex}(N = 200)$, is plotted as a function of kicking period T . For stronger kicks (dotted line; $K = 1, T < 10$) resonances proliferate and there is instability over almost all the parameter range. (b) Time-dependence near the (1, 1) resonance at $T \approx 10$ corresponding to Fig (a). Non-resonant ($T = 13$) curve shows weak quasi-periodic oscillations in N_{ex} ; the near-resonant regime, $T = 10$ is characterized by slow, large oscillations; at resonance $T = 10.5$, there is exponential growth in $N_{ex}(t)$. Inset shows that the condensate energy (calculated from the GPE itself) has similar oscillations.

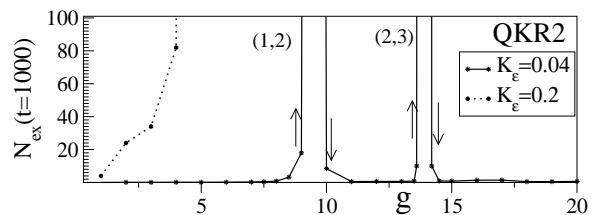


FIG. 2: double-kicked BEC (QKR2): Shows zones of instability occur at Bogoliubov resonances. Condensate losses as a function of nonlinearity parameter g . “Up” arrows indicate onset of exponential instability; “down” arrows means stability is regained. $N_{ex}(t = 1000)$ is plotted as a function of g (for $T = 2$, $\epsilon = 1/25$) for weak kicks ($K = 1$ so effective kick is $K_\epsilon = 0.04$) and stronger kicks ($K = 5$ so effective kick $K_\epsilon = 0.2$).

condensate and $\delta\psi$ represent the excited components. Inserting this form in the GPE and linearizing with respect to $\delta\psi$, we can write:

$$i\hbar \frac{d}{dt} \begin{pmatrix} \delta\psi \\ \delta\psi^* \end{pmatrix} = \mathcal{L}(t) \begin{pmatrix} \delta\psi \\ \delta\psi^* \end{pmatrix}. \quad (5)$$

where,

$$\mathcal{L}(t) = \begin{pmatrix} H(t) + g|\psi|^2 & g\psi^{*2} \\ -g\psi^{*2} & -H(t) - g|\psi|^2 \end{pmatrix}. \quad (6)$$

The analysis of condensate stability for a time-periodic system [16] reduces to the analysis of the operator $\mathcal{L}(t)$ over one period T . In general, for systems like BECs in modulated optical lattices, inter-mode coupling requires a detailed analysis of the instantaneous evolution. The nature of the δ -kicked potential permits considerable simplification.

The effect of $\mathcal{L}(t)$ reduces to the free-ringing of the eigenmodes of the unperturbed condensate for period T , interspersed by instantaneous impulses which mix the modes. Even for an experiment (where the kicks are approximated pulses of very short, but finite duration) numerical time-propagation is avoided: intermode coupling occurs over a very short time-scale, during which eigenmode phases remain essentially constant.

Excluding the kick term for the moment, we recall that the time propagation under H_{GP} can be analyzed in terms of the eigenmodes ($u_k(t), v_k(t)$) and eigenvalues of $\omega_k(t)$ of the 2×2 matrix on the right hand side of Eq.6. Setting $\psi = 1/\sqrt{2\pi}$, the matrix can be diagonalized and there are well-known analytical expressions for the unperturbed eigenmodes [1]

$$(u_k(t=0), v_k(t=0)) = \begin{pmatrix} U_k \\ V_k \end{pmatrix} \frac{e^{ik\theta}}{\sqrt{2\pi}}, \quad (7)$$

where $U_k + V_k = A_k$, $U_k - V_k = A_k^{-1}$, and $A_k = \left(\frac{\hbar^2 k^2}{2} \left(\frac{\hbar^2 k^2}{2} + \frac{g}{\pi} \right) \right)^{1/4}$.

In order to understand the behavior at the resonances, we introduce below a simple model using the eigenmodes Eq.7 as a basis. Writing the small perturbation in this basis:

$$\begin{pmatrix} \delta\psi(t) \\ \delta\psi^*(t) \end{pmatrix} = \sum_k b_k(t) \begin{pmatrix} U_k \\ V_k \end{pmatrix} \frac{e^{ik\theta}}{\sqrt{2\pi}} + b_k^*(t) \begin{pmatrix} V_k \\ U_k \end{pmatrix} \frac{e^{-ik\theta}}{\sqrt{2\pi}}. \quad (8)$$

Neglecting the kick, evolving the modes from some initial time t_0 , each eigenmode (u_k, v_k) simply acquires a phase ie:

$$b_k(t) = b_k(t_0) e^{-i\omega_k(t-t_0)}, \quad (9)$$

where $\omega_k = \sqrt{\frac{k^2}{2} \left(\frac{\hbar^2 k^2}{2} + \frac{g}{\pi} \right)}$.

After a time interval T , a kick is applied which couples the eigenmodes. Its effect is obtained by expressing the perturbation in a momentum basis, $\psi = \sum_l a_l(t)|l\rangle$ where $|l\rangle = \frac{e^{il\theta}}{\sqrt{2\pi}}$, and we can restrict ourselves to the symmetric subspace $a_l = a_{-l}$ of the initial condensate (parity is conserved in our system). Then, we can see by inspection that

$$a_k(t) = U_k b_k(t) + V_k b_{-k}^*(t). \quad (10)$$

Note that $b_k = b_{-k}$ for this system. Conversely, the corresponding amplitude b_k in each eigenmode k is given trivially from Eq.8 using orthonormality of the momentum states and the relation $U_k^2 - V_k^2 = 1$, yielding

$$b_k(t) = U_k a_k(t) - V_k a_k^*(t). \quad (11)$$

If the evolving condensate is given in the momentum basis, the effect of a kick operator $U_{kick} = e^{\pm \frac{iK}{\hbar} \cos \theta}$ is well-known. The matrix elements:

$$U_{nl} = \langle n | U_{kick} | l \rangle = J_{n-l}(K/\hbar) i^{\pm(l-n)} \quad (12)$$

The J_{n-l} are Bessel functions.

The amplitudes $a_l(t)$ are given by

$$a_n(t^+) = \sum_l i^{\pm(l-n)} J_{n-l} \left(\frac{K}{\hbar} \right) a_l(t^-), \quad (13)$$

where $a_n(t^+)/a_l(t^-)$ denotes the amplitude in state $|n\rangle$ just after/before the kick.

We can now define a “time-evolution” operator $\mathcal{L}'(T) = \mathcal{B}^{-1} \mathcal{L}_{free}(T) \mathcal{B} U_{kick}$, where \mathcal{L}_{free} denotes free ringing of the eigenmodes, \mathcal{B} is the transformation from momentum basis to Bogoliubov basis and U_{kick} is the action of the kick. A usual procedure for stability analysis of a driven condensate is to examine the eigenvalues of $\mathcal{L}'(T)$ to ascertain whether there is one (or more eigenvalues) which have a real, positive component [16], ie whether they produce exponential growth in the amplitudes $a_{\pm l}$.

However, to compare with GPE numerics, we simply evolve the mode amplitudes in time over a few kicks and examine the overall condensate response to the kicking (in the limit of very weak kicking). Hence we can evolve the amplitudes $a_l(t = NT)$ of the condensate perturbation from period N to period $N + 1$:

$$\mathbf{a}((N+1)T) = \mathcal{L}'(T) \mathbf{a}(N), \quad (14)$$

using only the simple analytical coefficients in Eq.13 and Eq.9, provided we use the simple transformations in Eqs.10 and Eq.11 to switch between the Bogoliubov mode basis and the momentum basis. $\mathcal{L}'(T)$ is non-unitary, but the method is quantitative in the perturbative limit provided $\psi \simeq \psi_0$, ie we assume $a_0(N) = a_0(0) = 1$.

We calculate the average energy over the first few N kicks, $\langle E(N) \rangle = \frac{1}{N} \sum_{t=1}^N E(t)$. Slow, large amplitude oscillations in $E(t)$ yield a large $\langle E(t) \rangle$ and indicate a resonance. Fig.3(a) shows the QKR-BEC behavior, for equivalent parameters to Fig.1(a). For low $K = 0.2$, there is the same single (1, 1) resonance at $T \approx 10$ as in Fig.1(a). For higher $K = 1$ the method is far from quantitative: the model Eq.14 is only a valid means of time-evolving the perturbation over a few kicks for small $K \ll 1$ since it assumes the perturbed component is negligible; nevertheless, for $K = 1$ it illustrates the regime of dense, overlapping resonances.

In Fig.4(b) we compare the perturbative Eq.(14) results with full GPE numerics for the first 20 kick pairs of the QKR2 in the limit of weak kicks. It shows remarkably good agreement. Moreover the scaling of the resonances with K is well described. The QKR2 resonant Bogoliubov spectrum differs appreciably from the QKR case. Fig.4(b) shows that for QKR2, even for low $K = 1, \epsilon = 1/25$ ie $K_\epsilon \approx 0.04$ and low $g = 1$, both $l = 1$ and $l = 2$ resonances are strongly excited. The QKR2-BEC resonance intensity depends strongly on K : the $l = 2$ resonances scale as K^4 , while the $l = 1$ scaling is closer to K^2 . In the full GPE numerics, the position of the maxima depends slightly on K and g , but remains within a few percent of the unperturbed value, even for longer kicking times if K_ϵ remains small.

In the limit of weak driving, one can obtain explicit expressions for the condensate wavefunction as a function of time. We assume that $a_0 \approx 1/\sqrt{2\pi} \gg a_{l \neq 0}$. Then Eq.13 can be approximated by $a_n(t^+) \approx a_l(t^-) + U_{l0}/(\sqrt{2\pi})$. From Eq.11 and Eq.9 we see that the amplitude accumulated over a single period in each eigenmode is

$$b_l(N+1) = b_l(N) + (U_l U_{l0} - V_l U_{l0}^*) e^{i\omega_l T}. \quad (15)$$

Summing all contributions iteratively from $t = 0$, taking $b_l(0) = 0$, we obtain

$$b_l(N) = (U_l U_{l0} - V_l U_{l0}^*) \sum_{n=0}^{N-1} e^{in\omega_l T}, \quad (16)$$

and so for $\omega_l T \approx 2\pi$ all the contributions add in phase, analogously to the well-known (but unrelated) resonances of the non-interacting limit [11].

We can write $\sum_{n=0}^{N-1} e^{in\omega_l T} = e^{-i(N-1)\omega_l T} \Phi(\frac{N\omega_l T}{2})$ where the Φ function is:

$$\Phi\left(\frac{N\omega_l T}{2}\right) = \frac{\sin(N\omega_l T/2)}{\sin(\omega_l T/2)}. \quad (17)$$

We thus expect oscillations in each set of $\pm l$ momentum components of amplitude

$$|2a_l(N)|^2 \propto 4|U_{l0}|^2 \Phi^2\left(\frac{N\omega_l T}{2}\right). \quad (18)$$

Off-resonance there will be quasi-periodic oscillations (in e.g. the condensate energy) from the superposition of contributions characterized by different eigenfrequencies ω_l . Close to resonance, a single component dominates; if the l -th mode is resonant we can write $\omega_l T \approx 2\pi M + 2\delta$ where $2\delta \ll 1$ is the de-phasing from resonance. Then

$$|a_l(N)|^2 \propto \frac{|U_{l0}|^2}{\delta^2} \sin^2(N\delta), \quad (19)$$

and there are slow, periodic oscillations of large amplitude $\sim 4\frac{|U_{l0}|^2}{\delta^2}$, at a frequency δ which is *not* related to

any eigenmode frequency, but given rather by the de-phasing from resonance.

The QKR2 resonant excitation spectrum is rather different from the QKR, and is analysed further in the next section.

III: RESONANCES OF THE QKR2-BEC

In the limit $K_\epsilon \rightarrow 0$, we can obtain analytical expressions for the BEC wavefunction of the double-kicked system. Firstly note that when $g\epsilon \ll 1$, the non-linearity has little effect during the short time-interval ϵ . Using the relation,

$$e^{+i\frac{K}{\hbar} \cos \theta} e^{-ip^2 t \hbar / 2} e^{-i\frac{K}{\hbar} \cos(\theta)} = e^{-\frac{iK}{2\hbar} [\hat{p} + K \sin \theta]^2}, \quad (20)$$

the time evolution can be given as a ‘one-kick’ operator

$$\hat{U}(T) \approx U_{GP}^{(0)}(T, 0) e^{-\frac{iK}{2\hbar} [\hat{p} + K \sin \theta]^2}. \quad (21)$$

In the limit $p\epsilon \approx 0$, one can split the operators in Eq.21 and neglect a term $K \sin \theta \hat{p}$ to obtain the approximation

$$\hat{U}(T) \approx e^{-\frac{i}{2\hbar} \hat{p}^2 T} \cdot e^{-\frac{i}{\hbar} \left[\frac{K^2 \epsilon}{2} \sin^2 \theta - iK\epsilon \hbar \cos \theta \right]}, \quad (22)$$

leaving an effective single-kick quantum rotor with a kicking potential

$$V_{kick} = \left[\frac{K^2 \epsilon}{2} \sin^2 \theta - iK\epsilon \hbar \cos \theta \right] \sum_N \delta(t - NT). \quad (23)$$

The second term, curiously, appears as kicking potential with an imaginary, and \hbar dependent, kick strength $iK\hbar$. It is of purely quantum origin as it arises from the non-commutativity of p and $\sin \theta$, i.e.

$$iK\hbar \cos \theta = [K \sin \theta, \hat{p}]. \quad (24)$$

Nevertheless, as seen below, it is important for weak driving as it controls the amplitude of the first excited mode $l = \pm 1$.

The matrix elements of the modified kick V_{kick} , like those in Eq.13, are Bessel functions. Specifically, the effect of V_{kick} on the condensate amplitudes a_l is given by

$$a_n(t^+) = \sum_l U_{nl} a_l(t^-), \quad (25)$$

where $U_{nl} = \sum_m i^{n-l-m} J_m(\frac{K^2 \epsilon}{4\hbar}) J_{n-l-2m}(iK\epsilon)$, and $a_n(t^\pm)$ indicates momentum amplitudes before(-) and after(+) the kick, as in Eq.13. Since $K\epsilon \ll 1$ and $J_{|n|>1}(z) \simeq 0$, only Bessel functions of low order ($m = 0$ or 1) will be non-negligible, and we can use the small-argument approximations for them, namely $J_0(z) \approx 1$, $J_{\pm 1}(z) \approx \pm z/2$.

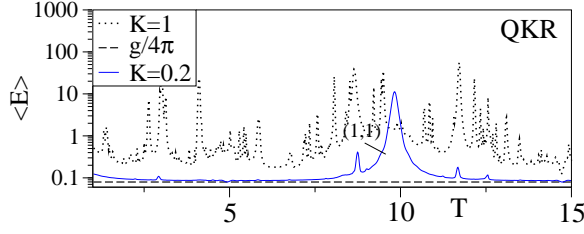


FIG. 3: Average energy $\langle E \rangle$ after 40 kicks. The dashed lines indicate the model of Eq.14; all other plots use full numerics. The label (n, l) denotes n -th resonance of mode l . Resonances of the QKR-BEC for parameters comparable to Fig.1a. For low $g = 1$, $K = 0.2$, only the single isolated $(1, 1)$ resonance is seen. For higher $K = 1$, resonances proliferate and overlap.

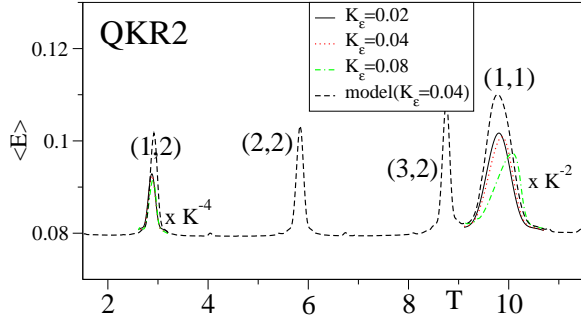


FIG. 4: Comparison between full GPE numerics and the model of Eq.14 for the QKR2-BEC, showing excellent agreement. Average energy $\langle E \rangle$ after 20 kick-pairs. The label (n, l) denotes n -th resonance of mode l . $g = 1$ and $\epsilon = 1/25$ so $K = 1$ corresponds to effective kick strength $K_\epsilon = 0.04$. For low K , the $l = 1$ resonance amplitudes scale as $\sim K^2$ while those of the $l = 2$ modes scale as $\sim K^4$.

Then, if the condensate is relatively unperturbed, the main effect of the kick will be to simply excite a small amount of $l = \pm 1$ and $l = \pm 2$ from the $|0\rangle$ state

$$e^{-\frac{i}{\hbar} V_{kick}} \psi \approx e^{-\frac{i}{\hbar} V_{kick}} |0\rangle = \sum_l U_{l0} |l\rangle \quad (26)$$

where

$$\begin{aligned} \sum_l U_{l0} |l\rangle &\approx \frac{1}{\sqrt{2\pi}} + i J_1 \left(\frac{i K \epsilon}{2} \right) |\pm 1\rangle + i J_1 \left(\frac{K^2 \epsilon}{4\hbar} \right) |\pm 2\rangle \\ &\approx \frac{1}{\sqrt{2\pi}} - \frac{K \epsilon}{4} |\pm 1\rangle + i \frac{K^2 \epsilon}{8\hbar} |\pm 2\rangle \end{aligned} \quad (27)$$

We obtain a similar equation to the QKR-BEC for the mode amplitudes, i.e.

$$b_l(N) = (U_l U_{l0} - V_l U_{l0}^*) \sum_{n=0}^{N-1} \exp[in\omega_l T].$$

But if only the lowest excited modes are significant, then, in particular,

$$\begin{aligned} b_1(N) &= -\frac{K\epsilon}{4} (U_1 - V_1) \sum_{n=0}^{N-1} \exp[in\omega_1 T] \text{ and} \\ b_2(N) &= i \frac{K^2 \epsilon}{8\hbar} (U_2 + V_2) \sum_{n=0}^{N-1} \exp[in\omega_2 T]. \end{aligned}$$

For $\omega_l T \approx 2\pi$ all the contributions add in phase and we will have a

resonance of either the $l = 1$ or $l = 2$ modes, the regime illustrated in Fig2(b).

Similarly as for the QKR-BEC, we can sum all the contributions to obtain an approximate analytical expression for the evolving condensate wavefunction including excited modes $l = \pm 1$ and $l = \pm 2$,

$$\psi(N) \approx \frac{1}{2\pi} [1 + C_1 \frac{K\epsilon}{2} \cos \theta + C_2 \frac{K^2 \epsilon}{4\hbar} \cos 2\theta]. \quad (28)$$

where

$$\begin{aligned} C_1 &= -\Phi(N\tilde{\omega}_1) [\cos(N-1)\tilde{\omega}_1 - i A_1^{-2} \sin(N-1)\tilde{\omega}_1], \\ C_2 &= \Phi(N\tilde{\omega}_2) [A_2^2 \sin(N-1)\tilde{\omega}_2 + i \cos(N-1)\tilde{\omega}_2], \end{aligned}$$

and $\tilde{\omega}_j = \omega_j T/2$.

Eq.28 shows that the amplitudes $|a_1|^2$ and $|a_2|^2$ scale as K^2 and K^4 respectively, as seen in the numerics in Fig.4(b). Fig.5(a) shows that Eq.28 gives excellent agreement with GPE numerics, giving accurately the non-resonant quasi-periodic condensate oscillations. Near the $l = 2$ resonance of Fig.2, Fig.5(b) confirms the QKR2 condensate oscillations (obtained from the GPE) scale quite accurately as $\propto \frac{1}{(\delta)^2} \sin^2 N\delta$ as expected from Eq.19 and Eq.28.

Fig.5(c) shows that, near-resonance, there are corresponding large oscillations in the non-condensate numbers calculated from Eq.4. Near-resonance, N_{ex} increases quadratically with time, on-resonance, the increase is exponential.

IV: BOGOLIUBOV RESONANCES FOR $T = 2\pi$

The kick period $T = 2\pi$, in a non-interacting system of cold atoms (i.e. $g = 0$) corresponds to a so-called “quantum anti-resonance” where the cold atom cloud exhibits periodic (period-2) oscillations. Hence the isolated Bogoliubov resonance regime to higher K than would be expected for generic T . The effect of a non-zero g for $T = 2\pi$ was investigated in [6]. An instability border occurring at a critical value of nonlinearity, e.g. for $g \simeq 2$ at $K = 0.8$, was identified where the growth on non-condensate particles with time became exponential.

In Fig.5(a) we investigate the behavior near critical g , for $K = 0.8$. We see that if a wider range of g is considered, the stability border is also a resonance: the condensate rapidly recovers stability after the instability border is passed. The condensate is exponentially unstable for $g \simeq 2 \rightarrow 2.6$, but is quite stable for both $g = 1.5$ and $g = 3$, as shown. Fig.5(b) shows oscillations in the condensate energy, as a function of time; a smoothed plot is also shown. For $g = 1.5$ and $g = 2.8$ (off-resonance) the smoothed plots are flat; for $g = 2.2$ and $g = 2.5$ (near-resonant), slow deep oscillations are apparent.

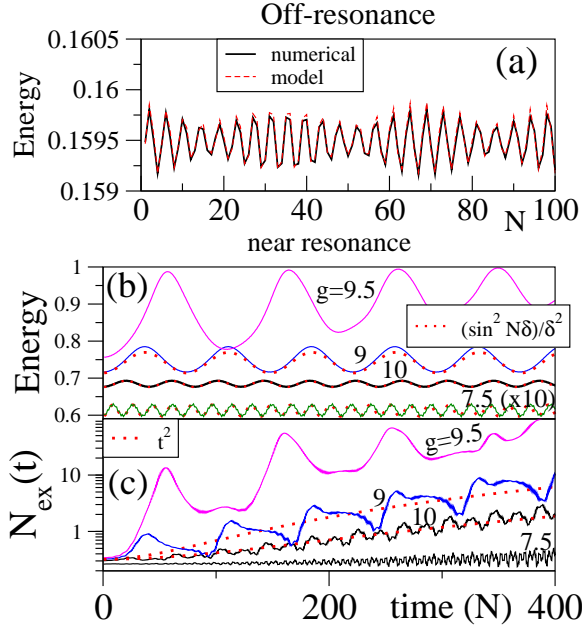


FIG. 5: Test of perturbative model. (a) Condensate energy oscillations from GPE numerics and Eq.28. $K_\epsilon = 0.04$, $g = 2$, $T = 2$. Beating between modes 1 and 2 is very accurately described by Eq.28. (b) Behavior of $l = 2$ resonance of Fig.1(b) $K_\epsilon = 0.04, T = 2$ and $g \approx 9.5$. As the resonance is approached the amplitude of the oscillations is proportional to the square of their wavelength, i.e. $E(t = NT) \propto K^4 \frac{1}{(\delta)^2} \sin^2 N\delta$ where 2δ is the distance from the resonance peak. (c) Corresponding number of non-condensate atoms from Eq.4.

The behavior is analogous to that of generic T ; however, the analysis of the condensate resonances for $T = 2\pi$ is less straightforward: the strongest resonances, even for low $K \lesssim 2$, do not in fact occur for $\omega_l T \approx 2\pi M$, where $M = 1, 2, 3, \dots$

A significant difference between generic T and $T = 2\pi$ is that, for the generic case, if we write

$$\omega_l T \approx 2\pi M_l + 2\delta(l) \quad (29)$$

we see that for arbitrary generic T , the distance from the nearest resonance, for the different modes, depends on l . In contrast, for $T = 2\pi$, for large l (i.e. $l \gtrsim 3$) we find $\omega_l T \approx (l^2 + \frac{g}{\pi})\pi$; in other words, the de-phasing from the nearest resonance (and hence the period of the mode oscillations) is similar (either $2\delta \approx \frac{g}{\pi}$ or $2\delta \approx 1 - \frac{g}{\pi}$) for all modes. So all mode oscillations for high l are approximately in phase with each other.

For $K = 0.8$, only low modes $l = 1, 2$ are significantly populated. These low modes ($l = 1$ and $l = 2$) are only in phase with each other at certain precise values of g, T . For these parameters, the model of Eq.14 predicts large resonances whenever the condition $(\omega_1 + \omega_2)T \approx 2\pi M$ is satisfied. In particular, for the resonance near $g \approx 2$, we find that for the $l = 1$ mode, $\omega_1 T \approx (1 - 2\delta)2\pi$ while for the $l = 2$ mode $\omega_2 T \approx (2 + 2\delta)2\pi$, with $2\delta \approx .25$.

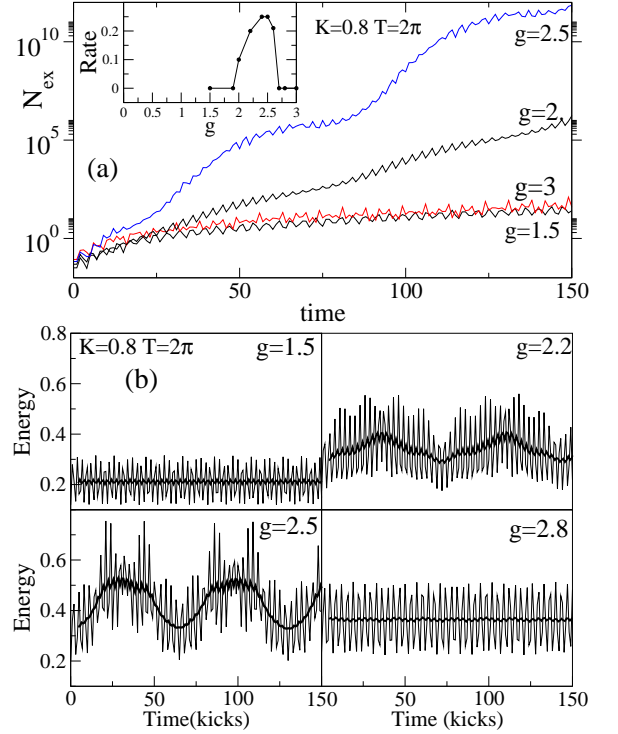


FIG. 6: (a) Non-condensate particles for kicking period $T = 2\pi$, $K = 0.8$, $g \approx 2$. The inset shows the rate of exponential growth of non-condensate atoms; zero denotes polynomial growth or less. The graph shows this instability border is a resonance: the condensate is unstable for $g = 2 - 2.5$ but is stable for $g = 1.5$ and $g = 3.0$. (b) Energy oscillations as a function of time; smoothed plots are also shown. Before and after the resonance ($g = 1.5$ and $g = 2.8$) the smoothed plots are flat. Near-resonance, ($g = 2.2$ and $g = 2.5$) the energy shows the characteristic slow, deep resonant oscillations.

These results suggest that “two-mode resonances”, i.e. synchronized oscillations of pairs of the lowest excited modes are the dominant mechanism for $T = 2\pi$ (NB this could be viewed as a “three-mode” resonance, if we include the lowest, initial mode, but $\omega_0 = 0$ for our system). They account for the shifting position of the critical instability border found by [6] in the $T = 2\pi$ case. For example, for slightly higher kick strengths, such as $K \simeq 2$, a resonance appears for $g \approx 1.65$ corresponding to $(\omega_2 + \omega_3)T \approx 2\pi M$, which accounts for the displacement of the instability border to lower values of g . Note that the resonance positions in the full numerics are K -dependent, whereas in the perturbative model of Eq.14 this dependence is neglected; the model is only valid for very small K .

V: CONCLUSION

In conclusion, we have shown that exponential instability in kicked BECs is related to parametric resonances, ie driving of low-lying collective modes at their natural frequencies, rather than to chaos in the underlying mean-field dynamics [24].

The signature of this process is in the onset of slow, large amplitude periodic oscillations in the condensate energy as well as the number of non-condensate atoms calculated from the time-dependent Bogoliubov formalism, as a resonance is approached. The resonances proliferate and overlap for large kick-strengths K , leading to instability over wider ranges of K and g . The time-dependent Bogoliubov approximation used here and in all other previous studies is only valid in regimes where the condensate depletion is negligible; for realistic condensates analysis of the dynamics in the narrow (for weak driving) windows of parametric instability, would require other approaches beyond Bogoliubov. However, away from these windows, the kicked condensate remains stable and relatively unperturbed, even after prolonged kicking.

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